

Exact solutions
of Reissner-Nordström type
in Einstein-Yang-Mills-systems
with arbitrary gauge groups
and space-time dimensions

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Abstract

We present a class of solutions in Einstein-Yang-Mills-systems with arbitrary gauge groups and space-time dimensions, which are symmetric under the action of the group of spatial rotations. Our approach is based on the dimensional reduction method for gauge and gravitational fields and relates symmetric EYM-solutions to certain solutions of two-dimensional Einstein-Yang-Mills-Higgs-dilaton theory. Application of this method to four-dimensional spherically symmetric (pseudo)riemannian space-time yields, in particular, new solutions describing both a magnetic and an electric charge in the center of a black hole. Moreover, we give an example of a solution with nonabelian gauge group in six-dimensional space-time. We also comment on the stability of the obtained solutions.

1 Introduction

The last few years were marked by an ever growing interest in solutions of Einstein-Yang-Mills-systems [1, 2, 3, 4, 5, 6, 7] and in the problem of their stability [8]. Most of these solutions were found numerically for the case of the gauge group $SU(2)$. In fact, the only known analytical solutions are the Reissner-Nordström-type solutions, which are effectively abelian. Such solutions were found for gauge groups $SU(2)$ and $SU(3)$ in [1, 7]. It is believed that such solutions can be easily constructed for arbitrary gauge groups from solutions of Einstein-Maxwell-systems by a procedure proposed in [9].

In this paper, we present a class of Reissner-Nordström-type solutions in EYM-theories with arbitrary gauge groups and in arbitrary spacetime dimensions, which are more general than those of ref. [9]. In fact, if the spacetime dimension is greater than four, our solutions can be nonabelian, in contrast to the solutions constructed in accordance with [9]. The specific structure of spherically symmetric four-dimensional spacetime stipulates that our solutions are in this case abelian. But nevertheless they are more general than those of ref. [9], because their effective gauge group is $U(1) \times U(1)$. In particular we find a new solution describing both a magnetic and an electric charge in the center of a black hole.

Our approach is based on the dimensional reduction method for gauge and gravitational fields and relates symmetric EYM-solutions in spacetime of arbitrary dimension to certain solutions of two dimensional Einstein-Yang-Mills-Higgs-dilaton (EYMHd) theory. This method essentially simplifies the problem of finding symmetric solutions, and we think that it can be useful for finding symmetric solutions of more general type. Moreover, relating the original system to a 2-d EYMHd-system gives the possibility to discuss some aspects of stability of these solutions in a quite transparent manner.

2 EYM-systems with arbitrary gauge group

We consider Einstein-Yang-Mills-theory with compact gauge group G on a spacetime E of the form

$$E = M \times K/H \quad , \quad (1)$$

where M is a contractible two dimensional manifold and K/H is an irreducible compact symmetric space (obviously, the sphere S^n is a special case of it). The group K is a simple compact Lie group, and its action on E is given by the canonical left action of K on K/H , we denote it by

$$\delta : K \times E \rightarrow E \quad . \quad (2)$$

Further we assume the existence of a lift

$$\sigma : K \times P \rightarrow P \quad (3)$$

of the action δ to automorphisms of the principal bundle $P(E, G)$.

A K -invariant EYM-configuration on E is given by a metric g on E and a connection form ω on $P(E, G)$ with

$$\delta_k^* g = g \quad \text{and} \quad (4)$$

$$\sigma_k^* \omega = \omega \quad \forall k \in K \quad . \quad (5)$$

First we note that the most general form of K -invariant metric on $E = M \times K/H$ is [15]

$$g = h \oplus e^{2\rho} \gamma_{K/H} \quad , \quad (6)$$

where h is an arbitrary metric on M , ρ is a function on M and $\gamma_{K/H}$ is a K -invariant metric on K/H .

If we fix a local coordinate system (x^μ) in M , the metric h can be written in the form

$$h = h_{\mu\nu} dx^\mu \overset{s}{\otimes} dx^\nu \quad . \quad (7)$$

Denoting $\dim K/H = n$, calculating the scalar curvature of E with this metric and omitting a complete divergence, we get

$$S_g = \frac{v_{K/H}}{16\pi\kappa} \int_M \left\{ R_M + e^{-2\rho} R_{K/H} + n(n-1) \partial_\mu \rho \partial^\mu \rho \right\} e^{n\rho} \sqrt{|h|} d^2 x \quad (8)$$

as the reduced gravitational action, where $v_{K/H}$ is the volume of the symmetric space K/H , calculated with the metric $\gamma_{K/H}$.

Next we consider a K -invariant connection form ω on $P(E, G)$. Bundles $P(E, G)$, admitting lifts (3) of the action (2), are in one-to-one correspondence with pairs (λ, \hat{P}) [10, 11], where

$$\lambda : H \rightarrow G \quad (9)$$

is a group homomorphism and \hat{P} is a principal bundle over M with reduced gauge group $C = C_G(\lambda(H))$ (centralizer of $\lambda(H)$ in G). A K -invariant connection form ω on $P(E, G)$ is in one-to-one correspondence [10, 12, 13, 14] with a pair $(\hat{\omega}, \Phi)$, where $\hat{\omega}$ is a connection form on \hat{P} and Φ is a mapping

$$\Phi : \hat{P} \rightarrow \mathfrak{M}^* \otimes \mathfrak{G} \quad , \quad (10)$$

satisfying

$$Ad\lambda(h) \circ \Phi(\hat{p}) = \Phi(\hat{p}) \circ Adh \quad , \quad \forall h \in H \quad . \quad (11)$$

Here \mathfrak{G} is the Lie algebra of G , and \mathfrak{M} is the complement of the Lie algebra \mathfrak{H} in the reductive decomposition $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{M}$.

To express the Yang-Mills action in terms of fields on M , we take a section s in the bundle \hat{P} and set $\hat{A} := s^* \hat{\omega}$, $\varphi := s^* \Phi$. Let $\{e_i\}$ be a basis of \mathfrak{M} ,

orthonormal with respect to $\gamma_{K/H}$. The pure Yang-Mills action on E reduces due to K -invariance to the following action on M

$$S_{YM} = \frac{v_{K/H}}{g^2} \int_M \left(\langle \hat{F}_{\mu\nu}, \hat{F}^{\mu\nu} \rangle + 2e^{-2\rho} \sum_i \langle D_\mu \varphi(e_i), D^\mu \varphi(e_i) \rangle - e^{-4\rho} V(\varphi) \right) e^{n\rho} \sqrt{|h|} d^2x \quad (12)$$

Here \langle, \rangle is the canonically normalized Ad -invariant bilinear form on \mathfrak{G} , \hat{F} is the curvature of \hat{A} , $D_\mu \varphi(e_i) = \partial_\mu \varphi(e_i) + [\hat{A}_\mu, \varphi(e_i)]$ is the covariant derivative of φ with respect to \hat{A} , and the scalar field potential is given by

$$V(\varphi) = - \sum_{k,l} \langle F_{kl}, F_{kl} \rangle, \quad (13)$$

$$F_{kl} = [\varphi(e_k), \varphi(e_l)] - \varphi([e_k, e_l] \downarrow \mathfrak{M}) - \lambda'([e_k, e_l] \downarrow \mathfrak{H}). \quad (14)$$

Obviously the second term of (14) vanishes for symmetric spaces K/H , and one can prove that $V(\varphi)$ in this case is a Higgs potential [10].

Thus, the reduced action of the full EYM-system is:

$$S = \int_M \left[\frac{1}{g^2} (\langle \hat{F}_{\mu\nu}, \hat{F}^{\mu\nu} \rangle + 2e^{-2\rho} \langle D_\mu \varphi, D^\mu \varphi \rangle - e^{-4\rho} V(\varphi)) + \frac{1}{16\pi\kappa} (R_M + e^{-2\rho} R_{K/H} + n(n-1) \partial_\mu \rho \partial^\mu \rho) \right] e^{n\rho} \sqrt{|h|} d^2x \quad (15)$$

Clearly, this is the action of 2-dimensional dilaton gravity coupled to gauge and Higgs fields.

We are interested in K -symmetric solutions of EYM-theory. One can prove that there is a one-to-one correspondence between K -symmetric solutions of EYM-theory and solutions of the field equations resulting from the reduced action [15, 16]. Therefore, it is sufficient to solve the field equations of the reduced theory. These equations are obtained by the variation of the reduced action with respect to the field variables. Note that we have to vary the action (15) before choosing particular coordinates, otherwise we lose some field equations.

We set $x^1 = r = \exp \rho$ and choose $x^0 = t$ in such a way, that the metric tensor $(h_{\mu\nu})$ is diagonal. Let $\alpha^2 = \varepsilon h_{00}$ and $\beta^2 = h_{11}$, where the variable ε has value 1 in Euclidean and value -1 in Minkowski spacetime. Then the resulting equations are

$$0 = g^2 \frac{\delta S}{\delta \varphi(e_l)} = -4r^{(n-4)} \alpha \beta \sum_k [\varphi(e_k), [\varphi(e_k), \varphi(e_l)] - \lambda'([e_k, e_l])] - 4D_\mu (\alpha \beta r^{(n-2)} D^\mu \varphi(e_l)) \quad (16)$$

$$0 = g^2 \frac{\delta S}{\delta \hat{A}_\nu} = -4D_\mu (\alpha \beta r^n \hat{F}^{\mu\nu}) + 4\alpha \beta r^n \sum_k [\varphi(e_k), D^\nu \varphi(e_k)] \quad (17)$$

$$0 = \frac{1}{\alpha\beta} \frac{\delta S}{\delta h^{01}} = \frac{1}{16\pi\kappa} \frac{\partial_0\beta}{\beta} n r^{(n-1)} + \frac{2}{g^2} r^{(n-2)} \langle D_0\varphi, D_1\varphi \rangle \quad (18)$$

$$0 = \frac{1}{\alpha\beta} \frac{\delta S}{\delta e^\rho} = \frac{1}{16\pi\kappa} \left[n r^{(n-1)} R_M + (n-2) r^{(n-3)} R_{K/H} \right. \\ \left. - n(n-1)(n-2) r^{(n-3)} \frac{1}{\beta^2} - 2n(n-1) r^{(n-2)} \partial_1 \left(\frac{\alpha}{\beta} \right) \frac{1}{\alpha\beta} \right] + \\ \frac{1}{g^2} \left[n r^{(n-1)} \langle \hat{F}_{\mu\nu}, \hat{F}^{\mu\nu} \rangle + 2(n-2) r^{(n-3)} \langle D_\mu\varphi, D^\mu\varphi \rangle \right. \\ \left. - (n-4) r^{(n-5)} V(\varphi) \right] \quad (19)$$

$$0 = \frac{\delta S}{\delta \alpha} = \frac{1}{16\pi\kappa} \left[+ 2n r^{(n-1)} \frac{\partial_1\beta}{\beta^2} + r^{(n-2)} \beta R_{K/H} - n(n-1) r^{(n-2)} \frac{1}{\beta} \right] \\ + \frac{1}{g^2} \left[- 2 \frac{\varepsilon}{\alpha^2\beta} r^n \langle \hat{F}_{01}, \hat{F}_{01} \rangle - \frac{2\varepsilon\beta}{\alpha^2} r^{(n-2)} \langle D_0\varphi, D_0\varphi \rangle \right. \\ \left. + \frac{2}{\beta} r^{(n-2)} \langle D_1\varphi, D_1\varphi \rangle - \beta r^{(n-4)} V(\varphi) \right] \quad (20)$$

$$0 = \frac{\delta S}{\delta \beta} = \frac{1}{16\pi\kappa} \left[- 2n r^{(n-1)} \frac{\partial_1\alpha}{\beta^2} + r^{(n-2)} \alpha R_{K/H} - n(n-1) r^{(n-2)} \frac{\alpha}{\beta^2} \right] \\ + \frac{1}{g^2} \left[- 2 \frac{\varepsilon}{\alpha\beta^2} r^n \langle \hat{F}_{01}, \hat{F}_{01} \rangle + \frac{2\varepsilon}{\alpha} r^{(n-2)} \langle D_0\varphi, D_0\varphi \rangle \right. \\ \left. - \frac{2\alpha}{\beta^2} r^{(n-2)} \langle D_1\varphi, D_1\varphi \rangle - \alpha r^{(n-4)} V(\varphi) \right] \quad (21)$$

These equations are difficult to solve in the general case. But they simplify considerably, if we restrict ourselves to solutions corresponding to constant *and* covariantly constant Higgs fields. In this case we have

$$d\varphi = 0 \quad \text{and} \quad [A_i, \varphi] = 0, \quad i = 0, 1 \quad . \quad (22)$$

Inserting (22) into equation (16) gives

$$\frac{\partial V}{\partial \varphi(e_l)} \equiv \sum_k [\varphi(e_k), [\varphi(e_k), \varphi(e_l)] - \lambda'([e_k, e_l])] = 0 \quad \forall l \quad . \quad (23)$$

Thus, the Higgs field configurations of such solutions are the extrema of the potential $V(\varphi)$. Equation (22) is, of course, the equation of spontaneous symmetry breaking for φ being in the Higgs vacuum.

Now we solve the obtained system of equations for this special case. Equations (20) and (21) give that $\alpha\beta$ is independent of r , and we can set

$$\alpha\beta = \eta(t) \quad . \quad (24)$$

Our next step is to solve the Yang-Mills-equations (17) in the special gauge $\hat{A}_1 = 0$. In this gauge we have

$$\hat{F}_{10} = \partial_1 \hat{A}_0$$

and we get for the zero component of (17)

$$0 = \partial_1 \left(\frac{r^n}{\alpha\beta} \hat{F}_{10} \right) = \partial_1 \left(\frac{r^n}{\alpha\beta} \partial_1 \hat{A}_0 \right) \quad . \quad (25)$$

Taking into account equation (24) we get after integration of (25):

$$\partial_1 \hat{A}_0 = \hat{F}_{10} = \frac{\mathcal{E}(t)\eta(t)}{r^n} \quad , \quad (26)$$

$$\hat{A}_0 = -\frac{\mathcal{E}(t)\eta(t)}{(n-1)r^{(n-1)}} + \mathcal{F}(t) \quad , \quad (27)$$

where $\mathcal{F}(t)$ and $\mathcal{E}(t)$ are time dependent elements of the Lie algebra \mathfrak{R} of the group R of the unbroken symmetry (which coincides with \mathfrak{C} , the centralizer of $\lambda'(\mathfrak{H})$ in \mathfrak{G} , if $\varphi = 0$). The remaining gauge freedom always allows us to put $\mathcal{F}(t) = 0$. Then the first component of the Yang-Mills-equation (17) leads to

$$\begin{aligned} 0 &= \partial_0 \left(\frac{r^n}{\alpha\beta} \hat{F}_{01} \right) + [\hat{A}_0, \hat{F}_{01}] \frac{r^n}{\alpha\beta} \\ &= \partial_0 (\mathcal{E}(t)) + \left[-\frac{\mathcal{E}(t)\eta(t)}{(n-1)r^{(n-1)}}, \frac{\mathcal{E}(t)\eta(t)}{r^n} \right] \frac{r^n}{\alpha\beta} \\ &= \partial_0 (\mathcal{E}(t)) \quad . \end{aligned} \quad (28)$$

This means that \mathcal{E} has to be a constant element of the Lie algebra \mathfrak{R} . Thus, the gauge field \hat{A} has the simple form

$$\hat{A} = -\frac{\eta(t)}{(n-1)r^{(n-1)}} \mathcal{E} dt \quad . \quad (29)$$

Another consequence is, that the *gauge-invariant* quantity

$$C := \frac{r^{2n}}{\alpha^2\beta^2} < \hat{F}_{01}, \hat{F}_{01} > = < \mathcal{E}, \mathcal{E} > \quad (30)$$

has to be constant. Taking into account equations (30) and (24) we can solve equation (20), which gives

$$\frac{nr^{(n-1)}}{\beta^2} = \frac{r^{(n-1)}}{n-1} R_{K/H} + \frac{16\pi\kappa}{g^2} \left(\frac{2\varepsilon C}{(n-1)r^{(n-1)}} - \frac{V(\tilde{\varphi})r^{(n-3)}}{n-3} \right) + D \quad , \quad (31)$$

where $V(\tilde{\varphi})$ is the value of the potential (13) in an extremum $\tilde{\varphi}$, and D is an integration constant. Because of equation (18), the quantity β has to be independent of t . Hence D has to be independent of t , too.

Thus, we arrive at the following result: for every solution of the EYM-system, corresponding to a constant and covariantly constant field $\tilde{\varphi}$, the metric tensor is given by

$$\beta^2 = \left(\frac{R_{K/H}}{n(n-1)} + \frac{16\pi\kappa}{g^2} \left[\frac{2\varepsilon C}{n(n-1)} r^{2(1-n)} - \frac{V(\tilde{\varphi})}{n(n-3)r^2} \right] + \frac{D}{nr^{(n-1)}} \right)^{-1}, \quad (32)$$

$$\alpha^2 = \frac{\eta(t)^2}{\beta^2}. \quad (33)$$

Differentiating equation (20) and using equation (24) one can easily see that our solutions also fulfill equation (19). Thus, we really have got a solution of the reduced EYM-system with the action (15). This solution can be of interest for 2-dimensional dilaton gravity. In fact, it is similar to the solution in 2-dimensional dilaton gravity coupled to abelian gauge field, which was found in [17].

Since we are interested in the solutions of the original theory, now we have to reconstruct the Yang-Mills gauge field A on spacetime $M \times K/H$. For this purpose, we define a section s_2 in the principal H -bundle $K \rightarrow K/H$. Let θ be the canonical left-invariant Lie-algebra-valued 1-form on the group K , $\bar{\theta} := s_2^* \theta$ its pull back, and $\bar{\theta}_{\mathfrak{H}}$, $\bar{\theta}_{\mathfrak{M}}$ the \mathfrak{H} - and \mathfrak{M} -components of the latter with respect to the reductive decomposition $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{M}$. Then the gauge field A on $M \times K/H$ is given by [10]

$$A = \hat{A} + \varphi(\bar{\theta}_{\mathfrak{M}}) + \lambda'(\bar{\theta}_{\mathfrak{H}}) \quad (34)$$

3 Examples

First we consider the four dimensional spherically symmetric space $E = M \times S^2$ with coordinates (t, r, ϑ, ϕ) . Since $S^2 = SU(2)/U(1)$, the space-time symmetry group K is $SU(2)$ and the stabilizer group H is $U(1)$. We parameterize $SU(2)$ by the Euler angles

$$SU(2) \ni u = \frac{\sqrt{2}}{2} (1 - i\sigma_2) \exp(i\frac{\phi}{2}\sigma_3) \exp(i\frac{\vartheta}{2}\sigma_2) \exp(i\frac{\psi}{2}\sigma_3) \quad , \quad (35)$$

$$0 \leq \phi < 2\pi, 0 \leq \vartheta \leq \pi, 0 \leq \psi \leq 4\pi, \quad (36)$$

where σ_i are the Pauli matrices. The group $H = U(1)$ is given by

$$H = \left\{ \exp(i\frac{\psi}{2}\sigma_3), 0 \leq \psi \leq 4\pi \right\}. \quad (37)$$

Hence, we have the reductive decomposition $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{M}$, with

$$\mathfrak{H} = \{z i \sigma_3; z \in \mathbb{R}\}, \quad (38)$$

$$\mathfrak{M} = \{x i \sigma_1 + y i \sigma_2; x, y \in \mathbb{R}\}. \quad (39)$$

If we define the section $s_2 : S^2 = SU(2)/U(1) \rightarrow SU(2)$ by

$$s_2(\vartheta, \phi) := \frac{\sqrt{2}}{2}(1 - i\sigma_2) \exp(i\frac{\phi}{2}\sigma_3) \exp(i\frac{\vartheta}{2}\sigma_2), \quad (40)$$

we get

$$s_2^*\theta = \frac{i}{2}\sigma_3 \cos \vartheta d\phi + \frac{i}{2}\sigma_2 d\vartheta + \frac{i}{2}\sigma_1 \sin \vartheta d\phi. \quad (41)$$

The invariant metric g on $E = M \times S^2$ is given by

$$g = \varepsilon \alpha^2 dt^2 + \beta^2 dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\phi^2). \quad (42)$$

We take the metric in the brackets as the standard metric $\gamma_{K/H}$ on S^2 . The scalar curvature $R_{K/H}$ of the symmetric space $S^2 = K/H$, calculated with this metric, is 2; the dimension n of K/H is obviously 2.

The action of $SU(2)$ on $S^2 = SU(2)/U(1)$ generates the following mapping Σ from \mathfrak{M} onto the tangent space $T_{(\vartheta=\pi/2, \phi=0)}S^2 = T_{[e]}SU(2)/U(1)$:

$$\Sigma(\frac{1}{2}i\sigma_1) = \partial_\phi, \quad (43)$$

$$\Sigma(\frac{1}{2}i\sigma_2) = \partial_\vartheta, \quad (44)$$

and we see, that $(e_1, e_2) = (i\sigma_1/2, i\sigma_2/2)$ is a basis of \mathfrak{M} , orthonormal with respect to $\gamma_{K/H}$.

Now we have to specify the gauge group G and the homomorphism $\lambda : H \rightarrow G$ respectively. We denote $\lambda' : \mathfrak{H} \rightarrow \mathfrak{G}$ the tangent mapping induced by λ . We choose $G = SU(4)$ as a simple example for demonstrating our method. Let $(\alpha_1, \alpha_2, \alpha_3)$ be a system of simple roots of $\mathfrak{G} = su(4)$. We map $i\sigma_3 \in \mathfrak{H}$ to $ih_{\alpha_1} + ih_{\alpha_3}$ by λ' , where h_{α_k} is the element of the Cartan subalgebra of $su(4)$ dual to the root α_k with respect to the canonically normalized bilinear form on \mathfrak{G} . One can easily see that the centralizer $\mathfrak{C}_{\lambda'(\mathfrak{H})}(\mathfrak{G})$ of $\lambda'(\mathfrak{H})$ in \mathfrak{G} consists of two $su(2)$ -subalgebras generated by the roots $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ and an abelian one-dimensional subalgebra spanned by the element $1/\sqrt{2}(h_{\alpha_1} + h_{\alpha_3})$ of the Cartan subalgebra. Thus, the gauge group of the reduced theory is $C = SU(2) \times SU(2) \times U(1)$.

To define the field φ , we have to solve equation (11) for the intertwining operator Φ . There are 4 complex parameters describing the constant Higgs field φ . We set

$$\varphi(\frac{1}{2}\sigma_1 + i\frac{1}{2}\sigma_2) = ae_{\alpha_1} + be_{\alpha_3} + ce_{-\alpha_2} + de_{\alpha_1+\alpha_2+\alpha_3}, \quad (45)$$

$$\varphi(\frac{1}{2}\sigma_1 - i\frac{1}{2}\sigma_2) = \bar{a}e_{-\alpha_1} + \bar{b}e_{-\alpha_3} + \bar{c}e_{\alpha_2} + \bar{d}e_{-\alpha_1-\alpha_2-\alpha_3}, \quad (46)$$

where $a, b, c, d \in \mathbb{C}$ and bar denotes complex conjugation. In accordance with equations (13) and (14), the potential V as a function of a, b, c, d takes the form

$$V = -2 \langle F_{\vartheta\phi}, F_{\vartheta\phi} \rangle \quad (47)$$

$$= \frac{1}{2}((a\bar{a} + c\bar{c} - 1)^2 + (a\bar{a} + d\bar{d} - 1)^2 + (b\bar{b} + c\bar{c} - 1)^2 + (b\bar{b} + d\bar{d} - 1)^2 + 2(a\bar{c} + \bar{b}d)(\bar{a}c + b\bar{d}) + 2(a\bar{d} + \bar{b}c)(\bar{a}d + b\bar{c})). \quad (48)$$

Now we have to look for the extrema of the potential V . We get 3 typical solutions: first, the trivial solution $a = b = c = d = 0$; second, the solution with $a = c = d = 0, b = 1$; and third, the solution with $a = b = 1, c = d = 0$.

In the first case $a = b = c = d = 0$, the field φ vanishes and the group R of the unbroken symmetry is the centralizer of $\lambda(H)$ in G , i.e. $SU(2) \times SU(2) \times U(1)$. The gauge field A on E is given by

$$\begin{aligned} A &= \hat{A} + \varphi(\bar{\theta}_{\mathfrak{M}}) + \lambda'(\bar{\theta}_{\mathfrak{H}}) \\ &= -\frac{\eta(t)}{r}\mathcal{E}dt + \frac{i}{2}(h_{\alpha_1} + h_{\alpha_3})\cos\vartheta d\phi \quad , \quad \mathcal{E} \in \mathfrak{R}. \end{aligned} \quad (49)$$

Obviously A is $u(1) \oplus u(1)$ valued. The potential V has the value 2.

In the second case $a = c = d = 0, b = 1$, the group of the unbroken symmetry is $U(1) \times U(1)$. The Lie algebra \mathfrak{R} is spanned by the elements h_{α_1} and $2h_{\alpha_2} + h_{\alpha_3}$ of the Cartan subalgebra of \mathfrak{G} . The gauge field A on E is given by

$$\begin{aligned} A &= \hat{A} + \varphi(\bar{\theta}_{\mathfrak{M}}) + \lambda'(\bar{\theta}_{\mathfrak{H}}) \\ &= -\frac{\eta(t)}{r}\mathcal{E}dt + \frac{i}{2}(h_{\alpha_1})\cos\vartheta d\phi + \left\{\frac{1}{2}(e_{\alpha_3} - e_{-\alpha_3})d\vartheta + \frac{i}{2}(e_{\alpha_3} + e_{-\alpha_3})\sin\vartheta d\phi + \frac{i}{2}(h_{\alpha_3})\cos\vartheta d\phi\right\} \quad , \quad \mathcal{E} \in \mathfrak{R} \quad . \end{aligned} \quad (50)$$

First observe that h_{α_1} and \mathcal{E} commute with the A_1 subalgebra spanned by $e_{\alpha_3}, e_{-\alpha_3}$ and h_{α_3} and, second, that the part in the braced brackets is a pure gauge. Therefore, we can choose a gauge such that the part in the braced brackets vanishes and the rest remains unchanged. Then we see that the gauge field is effectively $u(1) \oplus u(1)$ valued. The potential V has the value 1.

We can also calculate the first Chern number of the $U(1)$ connections on S^2 , giving the magnetic field component. For this purpose we have to consider the injective homomorphism $\tau : U(1) \rightarrow SU(4)$ defined by

$$\tau(e^{i\psi}) := e^{i\psi h} \in SU(4) \quad , \quad (51)$$

where $h = h_{\alpha_1} + h_{\alpha_2}$ in the first and $h = h_{\alpha_1}$ in the second case. Then we get

$$c = \frac{1}{2\pi i} \int_{S^2} (\tau')^{-1}(F) = \frac{1}{2\pi i} \int_{S^2} -\frac{i}{2} \sin\vartheta d\vartheta \wedge d\phi = -1 \quad . \quad (52)$$

Thus, we can interpret the S^2 part of the connection as a magnetic monopole with strength 1.

The third solution corresponds to $V = 0$, i.e. to the Higgs vacuum. In this case, the homomorphism $\lambda' : \mathfrak{H} \rightarrow \mathfrak{G}$ can be extended to a homomorphism from \mathfrak{K} to \mathfrak{G} . Hence, the field strength $F_{\vartheta\phi}$ vanishes, and we have only a $U(1)$ valued electric field

$$F = \frac{\eta(t)}{r^2} \mathcal{E} dr \wedge dt \quad , \quad (53)$$

with $\mathcal{E} \in \mathfrak{K}$. The group R of the unbroken symmetry in this case is $SU(2)$, and its Lie algebra \mathfrak{R} is spanned by $e_{\alpha_1+\alpha_2}+e_{\alpha_2+\alpha_3}$, $e_{-\alpha_1-\alpha_2}+e_{-\alpha_2-\alpha_3}$ and $h_{\alpha_1}+2h_{\alpha_2}+h_{\alpha_3}$. The fact that the solution corresponding to the Higgs vacuum of the reduced theory is a pure gauge on S^2 is valid for an arbitrary gauge group G . It is due to the specific structure of the spherically symmetric spacetime $E = M \times K/H$ in four dimensions, which includes the two-sphere $S^2 = SU(2)/U(1)$, and a general theorem [18], which says that any embedding of the Cartan element h_α of A_1 into an arbitrary Lie algebra can be extended to an embedding of the whole A_1 .

Now we can write down the metric coefficients α and β . If we set the integration constant $D = -4M$, they take the form

$$\alpha^2 = \eta(t)^2 \left\{ 1 + \frac{16\pi\kappa}{g^2} \frac{1}{r^2} \left(\frac{2\varepsilon < \mathcal{E}, \mathcal{E} >}{2} + \frac{V(\tilde{\varphi})}{2} \right) - \frac{2M}{r} \right\} \quad , \quad (54)$$

$$\beta^2 = \left\{ 1 + \frac{16\pi\kappa}{g^2} \frac{1}{r^2} \left(\frac{2\varepsilon < \mathcal{E}, \mathcal{E} >}{2} + \frac{V(\tilde{\varphi})}{2} \right) - \frac{2M}{r} \right\}^{-1} \quad . \quad (55)$$

We can interpret this solution in the first two cases as that of a magnetic and an electric charge in the center of a black hole with Schwarzschild radius $2M$, but in the third case we have only an electric charge in the center of the black hole. These solutions are similar to the well known Reissner-Nordström solution of Einstein-Maxwell-theory. We stress that our solutions can be easily found explicitly for arbitrary gauge groups G . One can show - using equation (22) - that our approach, independently of the gauge group G and the homomorphism λ , always gives commutative solutions, if the stabilizer group H is $U(1)$. Nevertheless, our solutions are more general than those of [9], because the field A in the first and second cases takes values in a two dimensional abelian subalgebra and not in a one-dimensional one, as in [9].

Next we give an example that our approach can also lead to noncommutative solutions. We consider the case of six-dimensional space-time $\mathbb{R}^2 \times K/H$, with

$$K/H = Sp(2)/(Sp(1) \times Sp(1)) \equiv SO(5)/SO(4) = S^4. \quad (56)$$

Then we have $K = Sp(2)$ and $H = Sp(1) \times Sp(1) = SU(2) \times SU(2)$. The embedding of the Lie algebra $su(2) \oplus su(2)$ into the Lie algebra $sp(2)$ is given by the decomposition of its defining representation $\underline{4}$:

$$\underline{4} \longrightarrow (\underline{2}, \underline{1}) + (\underline{1}, \underline{2}) \quad ,$$

i.e. if $\{e_\alpha, e_\mu\}, \alpha = 1, 2, \mu = 3, 4$ is a basis of $R_{\underline{4}}$, the representation space of $\underline{4}$, then the first $sp(1)$ acts on the space spanned by e_1, e_2 in the fundamental representation and trivially on the space spanned by e_3, e_4 , the second $sp(1)$ acts correspondingly on (e_3, e_4) in the fundamental representation and trivially on (e_1, e_2) . It is known that the adjoint representations of $sp(n)$ and $sl(n)$ can be expressed in terms of the their fundamental representations by [18]

$$ad\ sp(n) = \underline{(2n)} \overset{s}{\otimes} \underline{(2n)}, \quad (57)$$

$$ad\ sl(n) = \underline{n}^* \tilde{\otimes} \underline{n}, \quad (58)$$

where tilde means dropping out a one-dimensional trivial representation, \underline{n}^* denotes the contragradient representation, $t(x)^* = -t(x)^T$, and $\overset{s}{\otimes}$ denotes the symmetrized tensor product. Employing the method developed in [19], we can easily construct a basis of generators of $sp(2)$, adapted to the reduction to the $su(2) \oplus su(2)$ - subalgebra. This basis can be chosen to be $\{h_\beta^\alpha, h_\nu^\mu, d_{\alpha\mu}\}, \alpha, \beta = 1, 2, \mu, \nu = 3, 4$, with $\{h_\beta^\alpha = \varepsilon^{\alpha\gamma} e_\gamma \overset{s}{\otimes} e_\beta\}, h_\alpha^\alpha = 0; \{h_\nu^\mu = \varepsilon^{\mu\rho} e_\rho \overset{s}{\otimes} e_\nu\}, h_\mu^\mu = 0$ being the generators of $su(2) \oplus su(2)$ and $\{d_{\alpha\mu} = e_\alpha \overset{s}{\otimes} e_\mu\}$ being a basis of \mathfrak{M} . The antisymmetric tensor $\varepsilon_{ij} = \varepsilon^{ij}, i, j = 1, \dots, 4$ is defined by

$$\varepsilon_{12} = \varepsilon_{34} = 1, \quad \varepsilon_{13} = \varepsilon_{14} = \varepsilon_{23} = \varepsilon_{24} = 0.$$

We use the convention that the indices are raised by the second index and lowered by the first one, i.e. $\varepsilon^{\alpha\beta} d_{\beta\mu} = d^\beta_\mu$.

The generators of $sp(2)$ have the following commutation relations

$$\begin{aligned} [h_\beta^\alpha, h_\zeta^\gamma] &= \delta_\beta^\gamma h_\zeta^\alpha - \delta_\zeta^\alpha h_\beta^\gamma, \\ [h_\nu^\mu, h_\sigma^\rho] &= \delta_\nu^\rho h_\sigma^\mu - \delta_\sigma^\mu h_\nu^\rho, \\ [h_\beta^\alpha, d_{\gamma\mu}] &= -\delta_\gamma^\alpha d_{\beta\mu} + \frac{1}{2} \delta_\beta^\alpha d_{\gamma\mu}, \\ [h_\nu^\mu, d_{\alpha\rho}] &= -\delta_\rho^\mu d_{\alpha\nu} + \frac{1}{2} \delta_\nu^\mu d_{\alpha\rho}, \\ [d_{\alpha\mu}, d_{\beta\mu}] &= \frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon_{\rho\mu} h_\nu^\rho + \frac{1}{2} \varepsilon_{\mu\nu} \varepsilon_{\gamma\alpha} h_\beta^\gamma. \end{aligned} \quad (59)$$

and are normalized as

$$\begin{aligned} \langle h_\beta^\alpha, h_\zeta^\gamma \rangle &= \delta_\beta^\gamma \delta_\zeta^\alpha - \frac{1}{2} \delta_\beta^\alpha \delta_\zeta^\gamma, \\ \langle h_\nu^\mu, h_\sigma^\rho \rangle &= \delta_\nu^\rho \delta_\sigma^\mu - \frac{1}{2} \delta_\nu^\mu \delta_\sigma^\rho, \\ \langle d_{\alpha\mu}, d_{\beta\nu} \rangle &= -\frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon_{\mu\nu}. \end{aligned} \quad (60)$$

The representation $ad(\mathfrak{H})\mathfrak{M}$ is obviously $(\underline{2}, \underline{2})$.

Now we have to choose a gauge group and to define the homomorphism λ so that we get a nontrivial intertwining operator Φ , see equation (11). We take $\mathfrak{G} = A_{2m+1}$ and fix the embedding of $su(2) \oplus su(2)$ into it by the decomposition of the defining representation

$$\underline{2m+2} \longrightarrow (\underline{2}, \underline{1}) + m \times (\underline{1}, \underline{2}). \quad (61)$$

We take $\{f_\alpha, f_\mu \otimes g_s\}$, $\alpha = 1, 2$, $\mu = 3, 4$, $s = 1, \dots, m$ as an adapted basis of $R_{\underline{2m+2}}$, where $su(2) \oplus su(2)$ acts trivially on $\{g_s\}$ and on $\{f_\alpha, f_\mu\}$ analogously as $sp(2) \oplus sp(2)$ on $\{e_\alpha, e_\mu\}$ above. One can easily find that the centralizer of $su(2) \oplus su(2)$ in A_{2m+1} is $su(m) \oplus u(1)$, where $H \in u(1)$ acts on $\{f_\alpha\}$ by multiplication with $\frac{m}{m+1}$ and on $\{f_\mu \otimes g_s\}$ by multiplication with $-\frac{1}{m+1}$. The restriction of adA_{2m+1} to $su(2) \oplus su(2) \oplus su(m) \oplus u(1)$ reads

$$\begin{aligned} adA_{2m+1} &= (\underline{2m+2})^* \tilde{\otimes} (\underline{2m+2}) \longrightarrow \\ &(\underline{3}, \underline{1}, \underline{1})(0) + (\underline{1}, \underline{3}, \underline{1})(0) + (\underline{1}, \underline{1}, ad\, su(m))(0) + (\underline{1}, \underline{1}, \underline{1})(0) + \\ &+ (\underline{1}, \underline{3}, ad\, su(m))(0) + (\underline{2}, \underline{2}, \underline{m})(-1) + (\underline{2}, \underline{2}, \underline{m}^*)(1). \end{aligned} \quad (62)$$

A basis of generators in A_{2m+1} adapted to the decomposition (62) can be taken to be $\{H_\beta^\alpha, H_\nu^\mu, H_k^i, H, A_{\nu k}^{\mu i}, D_\alpha^{\mu i}, D_{\mu i}^\alpha\}$, $\alpha, \beta = 1, 2$, $\mu, \nu = 3, 4$, $i, k = 1, \dots, m$, and all the generators have zero trace in any pair of indices of the same kind. The generators

$$H_\beta^\alpha = f^\alpha \otimes f_\beta - \frac{1}{2} \delta_\beta^\alpha (f^1 \otimes f_1 + f^2 \otimes f_2) \quad (63)$$

$$H_\nu^\mu = (f^\mu \otimes g^i) \otimes (f_\nu \otimes g_i) - \frac{1}{2} \delta_\nu^\mu (f^\rho \otimes g^i) \otimes (f_\rho \otimes g_i) \quad (64)$$

span two $su(2)$ subalgebras, whereas the generators

$$H_k^i = (f^\rho \otimes g^i) \otimes (f_\rho \otimes g_k) - \frac{1}{m} \delta_k^i (f^\rho \otimes g^l) \otimes (f_\rho \otimes g_l) \quad (65)$$

span the $su(m)$; all of them have the same commutation relations as in (59). Another relevant commutator is

$$[D_\alpha^{\mu i}, D_{\nu k}^\beta] = \delta_\alpha^\beta A_{\nu k}^{\mu i} - \delta_\nu^\mu \delta_k^i H_\alpha^\beta + \frac{1}{2} \delta_\alpha^\beta \delta_\nu^\mu H_k^i + \frac{1}{m} \delta_\alpha^\beta \delta_k^i H_\nu^\mu - \frac{m+1}{2m} \delta_\alpha^\beta \delta_\nu^\mu \delta_k^i H. \quad (66)$$

We emphasize that the generators $\{H_\beta^\alpha, H_\nu^\mu, H_k^i\}$ are normalized as follows

$$\begin{aligned} \langle H_\beta^\alpha, H_\zeta^\gamma \rangle &= \delta_\beta^\gamma \delta_\zeta^\alpha - \frac{1}{2} \delta_\beta^\alpha \delta_\zeta^\gamma, \\ \langle H_\nu^\mu, H_\sigma^\rho \rangle &= m(\delta_\nu^\rho \delta_\sigma^\mu - \frac{1}{2} \delta_\nu^\mu \delta_\sigma^\rho), \\ \langle H_k^i, H_m^l \rangle &= 2(\delta_k^l \delta_m^i - \frac{1}{m} \delta_k^i \delta_m^l), \end{aligned} \quad (67)$$

which means that the second $su(2)$ -subalgebra has index m , and the $su(m)$ -subalgebra has index 2.

We see that the operator Φ intertwines $ad(\mathfrak{H})\mathfrak{M} = (\underline{2}, \underline{2})$ with $(\underline{2}, \underline{2}, \underline{m})(-1)$ and $(\underline{2}, \underline{2}, \underline{m}^*)(1)$ and is parameterized by a complex vector a^i , $i = 1, 2, \dots, m$.

Now we can define the homomorphism λ putting

$$\lambda'(h_\beta^\alpha) = H_\beta^\alpha, \quad \lambda'(h_\nu^\mu) = H_\nu^\mu. \quad (68)$$

Then Φ takes the form

$$\Phi(d_{\alpha\mu}) = \frac{1}{2}a_i\varepsilon_{\nu\mu}D_\alpha^{\nu i} + \frac{1}{2}a^i\varepsilon_{\beta\alpha}D_{\mu i}^\beta, \quad \bar{a}_i = a^i. \quad (69)$$

Before we calculate the potential V , see equation (13), we make some remarks about the resulting gauge field. If $m \geq 1$ and $|a|^2 = a_i a^i \neq 0$, the intertwining operator Φ maps \mathfrak{M} into the $su(4)$ subalgebra of $\mathfrak{G} = su(2m+2)$, spanned by

$$\tilde{H} = (f^\rho \otimes g^i) \otimes (f_\rho \otimes g_k) \frac{a_i a^k}{|a|^2} - f^\gamma \otimes f_\gamma, \quad H_\beta^\alpha, \quad (70)$$

$$\tilde{H}_\nu^\mu = \left((f^\mu \otimes g^i) \otimes (f_\nu \otimes g_k) - \frac{1}{2}\delta_\nu^\mu (f^\rho \otimes g^i) \otimes (f_\rho \otimes g_k) \right) \frac{a_i a^k}{|a|^2}, \quad (71)$$

$$D_\alpha^\mu = D_\alpha^{\mu i} \frac{a_i}{|a|}, \quad D_\nu^\beta = D_{\nu k}^\beta \frac{a^k}{|a|}. \quad (72)$$

$$(73)$$

If $|a|^2 = 1$, it extends the homomorphism τ' , defined by

$$\tau'(h_\beta^\alpha) = \lambda'(h_\beta^\alpha) = H_\beta^\alpha, \quad \tau'(h_\nu^\mu) = \tilde{H}_\nu^\mu \quad (74)$$

to a homomorphism from $sp(2)$ into $su(4)$. Hence, the field strength, see equation (14), takes values only in an $su(4) \oplus su(2)$ subalgebra of $su(2m+2)$, where the $su(2)$ subalgebra is given by $\tau'_1(\mathfrak{H})$, the homomorphism τ'_1 being defined by

$$\tau'_1(h_\nu^\mu) := H_\nu^\mu - \tilde{H}_\nu^\mu \quad (75)$$

$$= \left((f^\mu \otimes g^i) \otimes (f_\nu \otimes g_k) - \frac{1}{2}\delta_\nu^\mu (f^\rho \otimes g^i) \otimes (f_\rho \otimes g_k) \right) (\delta_i^k - a_i a^k),$$

$$\tau'_1(h_\beta^\alpha) := 0. \quad (76)$$

To calculate the potential V , we have to fix the standard invariant metric $\gamma_{K/H}$ on S^4 . We choose this metric so that the scalar product induced by it on \mathfrak{M} coincides up to the sign with the restriction to \mathfrak{M} of the canonically normalized invariant bilinear form \langle, \rangle on $sp(2)$.

Inserting the intertwining operator (69) into equation (13) and taking into account that the elements $\{(d_{\alpha\mu} + d^{\alpha\mu}), i(d_{\alpha\mu} - d^{\alpha\mu})\}$, $\alpha \leq \mu$, $\alpha < 2$ form an orthonormal basis of \mathfrak{M} , we get the following potential

$$V(\varphi) = 12(a_i a^i - 1)^2 + 6(m-1). \quad (77)$$

The first term of the potential comes from the $su(4)$ part of the gauge field and the second term comes from the $su(2)$ part.

The potential has two extrema: the absolute minimum (corresponding to the Higgs vacuum) with $a_i a^i = 1$ and the local maximum, for which $a_i = 0$.

First we will discuss the case $|a|^2 = a_i a^i = 1$. Having fixed a_i with this property, we see that the symmetry is broken to $R_0 = SU(m-1) \times U(1)$. The first term of the potential V vanishes, i.e. the $su(4)$ part of the gauge field is a pure gauge. Thus, we can choose a gauge so that this part of the gauge field vanishes and we arrive at the simple result that the gauge field on S^4 is $su(2) \equiv \tau_1'(\mathfrak{H})$ -valued. Hence, this solution corresponds in the framework of symmetric gauge fields to the homomorphism $\tau_1' : \mathfrak{H} \rightarrow \mathfrak{G}$ with vanishing intertwining operator Φ . The centralizer of $\tau_1'(\mathfrak{H})$ in $su(2m+2)$ is $\mathfrak{C}_{\tau_1'(\mathfrak{H})}(\mathfrak{G}) = su(m-1) \oplus su(4) \oplus u(1)$ and we have $\mathfrak{K} = \mathfrak{C}_{\tau_1'(\mathfrak{H})}(\mathfrak{G})$.

In the second case $a_i = 0, i = 1, \dots, m$ we have an $su(2) \oplus su(2) \equiv \lambda'(\mathfrak{H})$ -valued gauge field and $\mathfrak{K} = \mathfrak{C}_{\lambda'(\mathfrak{H})}(\mathfrak{G}) = su(m) \oplus u(1)$.

Thus, we have found the scalar field configurations, corresponding to the solutions of the original Yang-Mills equations. The explicit form of these solutions is given by formula (34), and it remains to find $\bar{\theta}_{\mathfrak{H}}$.

It is known that the group $K = Sp(2)$ can be represented by unitary quaternionic 2×2 -matrices. Then its Lie algebra \mathfrak{K} is parameterized as follows

$$\mathfrak{K} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & c \end{pmatrix}; a, b, c \in \mathbb{H}; \Re a = \Re c = 0 \right\}, \quad (78)$$

where \mathbb{H} is the skew field of quaternions, \bar{a} is the quaternionic conjugate and $\Re a$ is the real part of the quaternion a . The spaces \mathfrak{M} and \mathfrak{H} are given by

$$\mathfrak{M} = \left\{ \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}; b \in \mathbb{H} \right\}, \quad (79)$$

$$\mathfrak{H} = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}; a, c \in \mathbb{H}; \Re a = \Re c = 0 \right\}. \quad (80)$$

It is obvious that $[\mathfrak{H}, \mathfrak{M}] \subset \mathfrak{M}$ and $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{H}$. We identify the upper $su(2)$ -subalgebra with the one spanned by h_{β}^{α} and the lower $su(2)$ -subalgebra with the one spanned by h_{ν}^{μ} .

Now we have to construct a section $s_2 : K/H = S^4 \rightarrow K = Sp(2)$ to calculate the form $\bar{\theta} = s_2^* \theta$. We can parameterize the group $Sp(2)$ by $u_1, u_2 \in SU(2)$ and $x \in \mathbb{H}$:

$$\frac{1}{\sqrt{1+x\bar{x}}} \begin{pmatrix} 1 & x \\ -\bar{x} & 1 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u_2 \end{pmatrix} \in Sp(2). \quad (81)$$

This parameterization is not valid for infinite x , but that is clear because $K/H = S^4$. We identify $\mathbb{H} \cup \infty$ with S^4 and choose the section s_2 by putting

$$s_2(x) := \frac{1}{\sqrt{1+x\bar{x}}} \begin{pmatrix} 1 & x \\ -\bar{x} & 1 \end{pmatrix}. \quad (82)$$

After a short calculation we get

$$\begin{aligned}\bar{\theta} &= s_2^* \theta = (s_2(x))^{-1} d(s_2(x)) \\ &= \frac{1}{2(1+x\bar{x})} \begin{pmatrix} xd\bar{x} - dx\bar{x} & dx \\ -d\bar{x} & \bar{x}dx - d\bar{x}x \end{pmatrix},\end{aligned}\tag{83}$$

$$\bar{\theta}_{\mathfrak{H}} = \frac{1}{2(1+x\bar{x})} \begin{pmatrix} xd\bar{x} - dx\bar{x} & 0 \\ 0 & \bar{x}dx - d\bar{x}x \end{pmatrix}.\tag{84}$$

Taking into account the relation between quaternions and Pauli matrices, we see that the two $su(2)$ -components of $\bar{\theta}_{\mathfrak{H}}$ are the fundamental instanton resp. anti instanton connections on S^4 . The gauge field A , see equation (34), reads

$$A = -\frac{\eta(t)}{(n-1)r^{(n-1)}} dt + \psi'(\bar{\theta}_{\mathfrak{H}}),\tag{85}$$

where $\mathcal{E} \in \mathfrak{R}$ and $\psi' = \tau_1'$ in the first and $\psi' = \lambda'$ in the second case and we see that the S^4 part of the gauge field is an instanton like solution. In particular, if we put $m = 0$, the intertwining operator Φ is identically zero, and the centralizer of $\lambda'(\mathfrak{H})$ in \mathfrak{G} is trivial in this case, so that \hat{A} has to be zero, and the gauge field on $M \times S^4$ consists only of the instanton gauge field on S^4 .

Hence, our approach can also lead to noncommutative solutions of EYM-systems, if the stabilizer group is not $U(1)$.

One more remark is in order. It is easy to see that our method of finding solutions of EYM-equations makes it possible to discuss stability: It is clear that solutions corresponding to the local maximum of the Higgs potential are unstable. This means that the full solutions with dynamical gravitational field are also unstable, because the YM degrees of freedom can not be compensated by the gravitational degrees of freedom. On the other hand solutions corresponding to the Higgs vacuum can be considered as stable, if we treat gravitation as a fixed background field. But, of course, to discuss stability in the full dynamical context is very complicated. One can use for instance the concept of linear stability as discussed in [8].

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